A NUMERICAL STUDY OF MULTIDOMAIN DQMs FOR THE SOLUTION OF KdV EQUATIONS

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In Mathematics

By

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DECLARATION

I, H. Dipali Singh, hereby declare that the dissertation entitled "**A Numerical Study of Multidomain DQMs for the solution of KdV equations**" is an original work carried out by me under the guidance of Dr. Thoudam Roshan Singh. The contents of this dissertation did not form the basis of any previous degree to me or to the best of my knowledge, and that the dissertation has not been submitted by me for any research degree in any other university/institute. This is submitted to Sikkim University for the award degree of Master of Philosophy in Mathematics.

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A Numerical Study of Multidomain DQMs for the solution of KdV equations

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“Teachers can change lives with just the right mix of chalk and challenges".

Keeping this in mind, I feel it is a great opportunity for me to convey my heartfelt thanks and deepest appreciation through this small piece of acknowledgement to those precious persons.

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(H. Dipali Singh)

ABSTRACT

In our day-to-day life we encounter with different forms of wavesi. One such wave is shallow water waves which has the property of soliton. John Scott Russell was first to observe this phenomenon. In 1895, Diederik Korteweg and Gustav de-Vries derived the non-linear wave equation that depicts the shallow water waves. This non-linear differential equation is known as KdV equation. In this paper, we basically focused on finding the numerical solution of KdV equation and complex modified KdV equation. The paper is divided into six chapters. Chapter 1 contains the history of solitary waves and solitons. Brief introduction is given on KdV, differential quadrature method and multidomain. Chapter 2 has all necessary details about the KdV equation, its derivative, applications, etc. Chapter 3 includes different methods of differential quadrature for finding the weighting coefficients. In chapter 4, numerical experiments are carried out for KdV equations. Here, we considered two test problems to demonstrate the accuracy and efficiency of the proposed method. The results are presented in Tables and Figures. Chapter 5 include the numerical experiment for complex modified KdV equation. Here also we consider two test problems to show the accuracy and efficiency of our proposed method. Finally chapter 6 contains the conclusion part of this paper.

KEY WORDS: KdV equation, complex modified KdV equation, differential quadrature methods, multidomain, quintic B-spline

Contents

Chapter 1

INTRODUCTION

In our surrounding, we encounter different forms of waves like sound waves, water waves, tidal waves, magnetic waves, electric waves and many more. Their physical appearance along with their propagation has always interested researchers for last 150 years . Here, we have consider shallow water waves that appear on the canal water, occur due to small disturbance in the water surface.

1.1 History of Solition

The shallow water waves are the solitary waves that show the property of the solitons i.e they retain their shape even if they collide with other similar waves. The soliton phenomenon was first described in 1834 by John Scott Russell (1808 – 1882) who observed a solitary wave in the union canal in Scotland. He produced the phenomenon in a wave tank and named it the Wave of Translation. Here his original text as he described is as follows

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation"[1].

Russell did extensive experiments in his laboratory where he build a water tank to replicate the phenomenon in order to study more carefully. While doing so Russell have obtained the following results[26]

- Solitary waves are long shallow water waves of permanent form, hence he deduced that they exist, this is his most significant result.
- The speed of propagation c, of a solitary wave in a channel of uniform depth h is given by $c^2 = g(h + \eta)$, where η is the amplitude of the wave and g the force due to gravity.

However, the significance was seen after a century later. In an numerical study by Fermi, Pasta and Ulam (1955) they discovered soliton[2]. They were studying the heat transfer problem also known as FPU problem by a nonlinear springs connected with 1D lattice. They thought energy will be equally shared in lattice in the initial state, but no such thing was observed rather the system returned to its initial state recurrently. Later, in 1965 Zabusky and Kruskal looked into the FPU problem and derived an asymptotic description of oscillations of unidirectional[2, 3] waves propagation that reduced to KdV equation. Then, the numerical study shows the existence of types of solitary waves which behaves like particles just like photon, electron. So, they called such solitary waves as solitons.

1.2 Solitary Wave and Soliton

A solitary wave is a non-linear and localized wave which propagates without change of its properties(shape, velocity, etc)[26]. In 19th century, Solitary wave first came into notice in hydrodynamics[3, 4]. The name 'solitary wave' was suggested by John Scott Russell. It arises from a balance between non-linear and dispersive effects. Mostly in solitary waves the width depends on the amplitude.

Where as soliton is a solitary wave that behave like 'particle'. It was Zabusky and Kruskal[4] who named a solitary wave with the particle property as 'soliton'. Soliton represents permanent form of a wave. It is localized like a solitary wave so that it decays or approaches a constant value at infinity. It is stable and retains its shape even after collisions with other similar waves. Solitons are the solutions of many weakly, non-linear dispersive, partial differential equations describing physical systems.

1.3 FPU Recurrence

In 1954 , the phenomenon of recurrence in non-linear system was first observed by Fermi, Pasta and Ulam[8]. They studied the thermalization process of a solid[5]. The idea was to observe the non-linear interaction that leads to the energy equipartition between large number of degree of freedom in the mechanical chain[46]. Instead of the energy equipartition they observed that after a time a recurrence to the initial data was achieved. This recurrence phenomenon was known as Fermi-Pasta-Ulam(FPU) problem[8, 46]. There are many theoretical and experimental proves that shows the existence of the FPU recurrences. Few were discussed here. Ermoshin(et.al)[6], studied the quantum mechanical wave-packet revivals and the cases of recurrence in coupled and uncoupled oscillators were also demonstrated. Ruban[7], studied the two-dimensional free-surface potential flows of incompressible fluid over a constant depth and gravity. And showed the FPU recurrence in the system. Kuznestsov[8] does a numerical study on one-dimensional NLSE (Non-Linear Schrödinger Equation). They explained the FPU recurrence in NLSE and shown that the FPU recurrence takes place not only for condensate but for cnoidal wave too. Shmid and et.al studied the coupled blood pressure dynamics and heart electrical dynamics[9]. They demonstrated the FPU recurrence in electrical activity of the heart. FPU spectra shows different states of cardiovascular system. A comparative study proved that the FPU spectra can be useful for diagnostics and also for evaluation of the therapeutic arrangements results.

1.4 The KdV Equation

In sec(1.1), we discussed about the history of solitary wave which was described by Scott-Russell. Many investigations were undertaken by different mathematicians and physicists like by Airy (1845), Stokes (1847), Boussinesq (1871, 1872) and Rayleigh (1876) in an attempt to understand this phenomenon. Then in 1895, Diederik Korteweg and Gustav de-Vries derived a non-linear wave equation which is known as KdV equation that was named after them and was mathematically represented as[1],

$$u_x(x,t) + \xi u^p u_x(x,t) + \mu u_{xxx}(x,t) = 0 \tag{1.1}$$

where p, ξ and μ are the real positive parameters. The non-linear term $u^p u_x$ causes the steeping of the wave form and the dispersive term u_{xxx} make the wave form spread. Now, these two parts give rise to solitons, which represents waves of permanent form. Thus we can say that KdV equation is the most simplest form of non-linear partial differential equation that can be used in studying of solitary waves or solitons.

1.5 Differential Quadrature Method(DQM)

Now-a-days, Differential Quadrature Methods are widely used numerical approximation techniques to solve the initial and boundary value problems. When compared with other numerical methods like FDM(Finite Difference methods), FEM(Finite Element Methods) etc. Differential Quadrature Method(DQM) shows excellent numerical results in terms of accuracy and efficiency [10]. With few mesh points only we can get high-precise solutions, convergence rate is good and also it requires less

computational space[10, 11]. DQMs was first introduced by Bellman et al.(1971, 1972) [10]. Now, we will go into little detail of differential quadrature methods of different orders. Differential Quadrature Method is defined as approximation to derivatives of a function with respect to a coordinate direction which is expressed as a linear weighted sum of all the functional values at all the mesh points along that direction. The differential quadrature method was taken from the idea of integral quadrature. It was first proposed by Bellman and his associates in 1972.
The mathematical representation of differential quadrature[10] was given by

$$f_x^{(p)} = \sum_{j=1}^{N} W_{ij}^{(p)} f(x_j), i = 1, 2, 3, ..., N \tag{1.2}$$

where $f_x^{(p)}$ is the pth order derivative of the function f w.r.t the variable x and $W_{ij}^{(p)}$ are the weighting coefficients. The main work here is to determine the weighting coefficients which can be done by using function approximation methods. DQM is based on polynomial approximation because of which the differential quadrature relating to it known as Polynomial-Based Differential Quadrature (PDQ). This methods are used in analysing the numerical solutions of the time dependent partial differential equations. And out these methods most widely used methods are Lagrange based differential quadrature method and Fourier based differential quadrature methods.

1.6 Multi-domain Differential Quadrature Method

Since, differential quadrature approximation is based on polynomial approximations. So it is easy to carry out any problem of one-dimension using differential quadrature method. But, most of the engineering problems are of multi dimensions i.e two-dimensional or three-dimensional or more. So, we need to extend differential quadrature approximation from one-dimension to higher-dimension. In order to do so, Shu(1991) gave a suggestion that one-dimensional polynomial based differential quadrature can be extended to multi-dimensional forms if the domain is regular i.e a rectangular domain or spherical domain. Now, the multidomain technique[10] is used in which the computational domain is decomposed into several sub domains. Then each sub domain generated local grid points and local differential quadrature technique is applied in the same manner as in a single domain case. The neighbouring subdomains got the information through the interface.

We will go deep into each topic in the successive chapters and its involvement in our work

1.7 Objectives

- To approximate the numerical solutions of one-dimensional KdV equation and complex modified KdV equation.
- Using Differential Quadrature Method as the tool to solve the system of equations.
- To introduce multi-domain differential quadrature, to find the weighting coefficients.

Chapter 2

THE KORTEWEG-de VRIES(KdV)EQUATION

In this chapter we will discuss about the the KdV equation in detail. As we know John Scott Russell was first to describes the solitary wave in 1834. Later many investigation were done to understand the phenomenon, then in 1895 Diederik Korteweg and Gustav de-Vries gave the partial differential equation which depicts the model of the solitary wave[1, 26]. Here, the main focus is given on the derivation of KdV equation, few conservation laws, area of applications, its different forms and theoretical review of Korteweg-de Vries equation.

2.1 Derivation of KdV Equation

In this section, we presented the derivation of KdV equation. We considered a two dimensional hydrodynamic wave problem. In figure 2.1 , we can see a solitary wave is moving along the x-axis. At rest, height of water is h, the height of solitary wave is a and the length of solitary wave is l. From the following assumptions we can get a wave equation of KdV type.

- A 2-dimensional problem, independent of z-axis.
- Mass density of water is constant.
- Irrotational flow of water.
- Boundary conditions:
    1. No flow of water across the river bed.
    2. No flow of water across the free surface.
    3. Pressure should be constant at the free surface.
- Neglecting the viscosity.
- 1. If $h \ll l$
    2. If $a \ll h$
- The waves move in one direction i.e along the x-axis.

By considering our first assumption, the velocity of the fluid can presented in the form of

$$\overrightarrow{v}(x, y, t) = \begin{pmatrix} u(x, y, t) \\ v(x, y, t) \end{pmatrix}$$

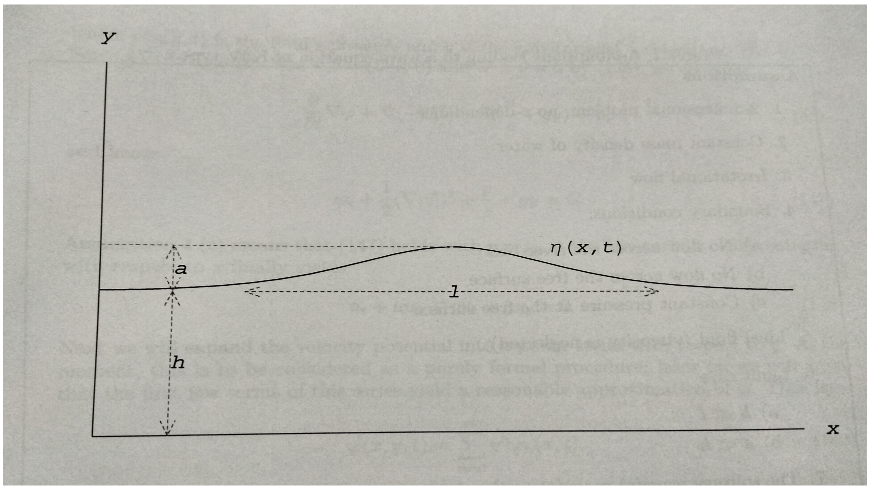

Figure 2.1: Schematic view of geometry of KdV equation

From our second assumption, the equation of mass conversation

$$\frac{\partial p}{\partial t} + \nabla.(\rho \overrightarrow{v}) = 0$$

can be simplified as

$$0 = \nabla.\overrightarrow{v} = u_x + v_y \tag{2.1}$$

From our third assumption, the velocity potential is given as

$$\begin{gathered} \nabla \times \overrightarrow{v} = \overrightarrow{0} \Rightarrow \overrightarrow{v} = \nabla\varphi \\ u = \varphi_x, \ \ v = \varphi_y \end{gathered} \tag{2.2}$$

From eqn(2.1) and eqn(2.2) we get the Laplace's equation

$$\Delta\varphi = \varphi_{xx} + \varphi_{yy} = 0 \tag{2.3}$$

The above equation gives a nonlinear equation at its boundary. From fourth(1) assumption, $\forall$ x and t the velocity component v vanishes at $y = 0$ for all x and t, we get

$$v(x,0,t) = \varphi_y(x,0,t) = 0 \tag{2.4}$$

The free water surface whose boundary condition is not fixed but evolves in time,and so it is called free boundary problem. The equation for free water surface is given by

$$y = \eta(x,t) \equiv h + q(x,t)$$

Again from fourth(2) assumption mass element remains constant, hence the coordinate is given by

$$y(t) = \eta(x(t),t)$$

Taking t-derivatives we get,

$$v = \frac{dy}{dt} = \frac{\partial \eta}{\partial x}u + \frac{\partial \eta}{\partial t}$$

or

$$\varphi_y = \frac{\partial \eta}{\partial x}\varphi_x + \frac{\partial \eta}{\partial t} \tag{2.5}$$

Last boundary condition i.e the fourth(3) assumption where pressure p is kept constant gives the conservation of momentum which is called as Navier-Stokes's equations or the Euler's equation for an ideal fluids given as

$$\frac{d\overrightarrow{v}}{dt} \equiv \frac{\partial \overrightarrow{v}}{\partial t} + (\overrightarrow{v}.\nabla)\overrightarrow{v} = -\frac{1}{\rho}\nabla p - \begin{pmatrix} 0 \\ g \end{pmatrix}$$

where p is the fluids's and g is the gravitational acceleration.

As $\frac{1}{2}\nabla(\overrightarrow{v}.\overrightarrow{v}) = (\overrightarrow{v}.\nabla)\overrightarrow{v} + \overrightarrow{v} \times (\nabla \times \overrightarrow{v})$ and from eqn(2.2), we get

$$\frac{\partial}{\partial t}\nabla\varphi + \nabla(\frac{1}{2}|\overrightarrow{v}|^2) + \nabla(\frac{p}{\rho} + gy) = \overrightarrow{0}$$

and hence

$$\varphi_t + \frac{1}{2}(\nabla| \to v|)^2 + \frac{p}{q} + gy = C$$

From fourth(3) assumption we take $p = p_0$ at the free surface and differentiating w.r.t x, we get

$$u_t + uu_x + vv_x + g\eta_x = 0 \tag{2.6}$$

Let us expand the velocity potential into a power series w.r.t y,

$$\varphi(x, y, t) = \Sigma_{n=0}^{\infty} y^n \varphi_n(x, t)$$

Since in eqn(2.3) the velocity potential satisfies Laplace's equation, hence we get

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\varphi &= \Sigma_{n=0}^{\infty} y^n \frac{\partial^2}{\partial x^2}\varphi_n(x, t) \\ \frac{\partial^2}{\partial x^2}\varphi &= \Sigma_{n=2}^{\infty} n(n-1)y^{n-2}\varphi_n(x, t) \\ &= \Sigma_{n=0}^{\infty}(n+2)(n+1)y^n\varphi_{n+2}(x, t) \end{aligned}$$

For φ_n where $n \epsilon N$, we have the recurrence relation as

$$\frac{\partial^2}{\partial x^2}\varphi_n(x, t) + (n+2)(n+1)\varphi_{n+2}(x, t) = 0$$

From eqn(2.4) the boundary condition gives

$$0 = \frac{\partial}{\partial y}\varphi(x, y, t)|_{y=0} = \Sigma_{n=1}^{\infty} n y^{n-1}\varphi_n(x, t)|_{y=0} = \varphi_1(x, t)$$

Hence from the above equation, we get

$$n = 0 : \frac{\partial^2}{\partial x^2}\varphi_0 + 2.1.\varphi_2 = 0$$

$$n = 1 : \frac{\partial^2}{\partial x^2}\varphi_1 + 3.2.\varphi_3 = 0$$

$$\Rightarrow \varphi_3(x, t) = 0$$

$$n = 2 : \frac{\partial^2}{\partial x^2}\varphi_2 + 4.3.\varphi_4 = 0$$

$$n = 3 : \frac{\partial^2}{\partial x^2}\varphi_3 + 5.4.\varphi_5 = 0$$

$$\Rightarrow \varphi_5(x, t) = 0$$

Thus all odd φ_n vanishes and the even φ_n can be expressed as even x-derivatives of φ_0 i.e

$$\varphi(x, y, t) = \Sigma_{n=0}^{\infty} y^{2n}\frac{(-1)^n}{(2n)!}(\frac{\partial}{\partial x})^{2n}\varphi_0(x, t) \tag{2.7}$$

From sixth assumption we considered the parameters

$$\alpha \equiv \frac{a}{h} \ll 1, \beta \equiv \frac{h^2}{l^2} \ll 1$$

which are assumed to be magnitude of same order and analyze the problem with asymptotic limit $\alpha, \beta \rightarrow 0$. Now transforming the variables and equations into a dimensionless form, we denote all the variables into small letters and substituting as

$$x \to lx$$
$$y \to hx$$
$$t \to \frac{l}{c}t$$
$$q \to aq$$
$$\varphi \to \frac{gla}{c}\varphi$$

where

$$c \equiv \sqrt{gh}$$

denotes a typical velocity. It can be shown equal to the velocity of linear water waves in the lowest order of approximation. The units which we choose are different because of two dimensionless quantities α and β. So, choosing any other units except the present ones we get different asymptotic limit of the wave equation. So, after the transformation, the dimensionless equation of the free surface is given by

$$y = \eta(x,t) = 1 + \alpha q(x,t) \tag{2.8}$$

Eqn(2.5) equals to

$$\varphi_y = q_x\varphi_x + q_t$$

Substituting the variables we get,

$$\frac{gla}{c}\frac{1}{h}\varphi_y = \frac{a}{l}\frac{gla}{c}\frac{1}{l}q_x\varphi_x + \frac{ac}{l}q_t$$

Multiplying $\frac{l}{ac}$, we get

$$\frac{1}{\beta}\varphi_y = \alpha q_x \varphi_x + q_t \tag{2.9}$$

Transforming eqn(2.6) into dimensionless form , we have

$$\varphi xt + \alpha\varphi_x\varphi xx + \frac{\alpha}{\beta}\varphi_y\varphi yx + q_x = 0 \tag{2.10}$$

The dimensionless form of the power series expansion of φ in eqn(2.7), we have

$$\varphi(x, y, t) = \Sigma_{n=0}^{\infty} y^{2n}\frac{(-1)^n}{(2n)!}\beta^n(\frac{\partial}{\partial x})^{2n}\varphi_0(x, t) \tag{2.11}$$

$$= \varphi_0(x, t) - \frac{\beta}{2}y^2\varphi_{0xx}(x, t) + \frac{\beta^2}{24}y_4\varphi_{0xxxx}(x, t) \mp \cdots \tag{2.12}$$

Let $w(x, t) \equiv \varphi_{0x}(x, t)$. Substituting eqn(2.12) in eqn(2.9) and eqn(2.10) and by using eqn(2.8), we have the asymptotic form

$$w_x + q_t + \alpha(qw)_x - \frac{\beta}{6}w_{xxx} = O_2 \tag{2.13}$$

$$w_t + q_x + \alpha w(w)_x - \frac{\beta}{2}w_{xxt} = O_2 \tag{2.14}$$

where O_2 is term for quadratic and higher order in α and β. considering the constant terms only, eqn2.13 and eqn(2.14) reduces to

$$w_x = -q_t, w_t = -q_x$$

Differentiating above equation we get w.r.t x we get

$$w_{xx} = -q_{tx} = -q_{xt} = w_{tt} \tag{2.15}$$

and for q

$$q_{xx} = q_{tt} \tag{2.16}$$

These are 1-dimensional linear wave equations for w and q with dimensionless wave velocity 1. This confirms the physical meaning of typical velocity c. The solutions of eqn(2.15) and eqn(2.16) are known as travelling wave trains which are of the form

$$w(x,t) = f(x \pm t), q(x,t) = F(x \pm t)$$

with arbitrary functions f and F. From seventh assumption q is travelling to right, so $q(x,t) = F(x-t)$. As

$$w_x = f'(x \pm t) = -q_t = F'(x-t)$$
$$w_t = \pm f'(x \pm t) = -q_x = -F'(x-t)$$

By our assumption, the consistent solutions of eqn(2.15) and eqn(2.16) are of the form

$$w(x,t) = f(x-t), f(\xi) = F(\xi) + C$$

the constant C vanishes as q and w vanishes for $|\xi| \to \infty$, so we have

$$w(x,t) = q(x,t) = F(x-t) \tag{2.17}$$

Eqn(2.13) and eqn(2.14) represents two coupled nonlinear partial differential equations. By using $q_{xt} = q_{tx}$ we can decoupled these equations but it leads to second order time derivative whereas KdV is of first order. So, we substitute eqn(2.17) in eqn(2.13) and eqn(2.14). For which we can use $w = q$ and $w_t = -w_x$ in the first order terms of the difference between eqn(2.13) and eqn(2.14) to get a more symmetric relation of the form

$$w_t - q_t + \frac{\alpha}{2}qq_t - \frac{\beta}{3}q_{xxt} = w_x - q_x + \frac{\alpha}{2}qq_x - \frac{\beta}{3}q_{xxx} \tag{2.18}$$

here we replaced the term O_2 with 0. We get the equation of the form

$$\frac{\partial}{\partial x}G(x,t) = \frac{\partial}{\partial x}G(x,t)$$

where

$$G(x,t) = w - q + \frac{\alpha}{4}q^2 - \frac{\beta}{3}q_{xx}$$

Let the general solution be $G(x,t) = g(x+t)$. As the difference between $w - q$ is of linear order $O(\alpha, \beta)$, we can write

$$w = q - \frac{\alpha}{4}q^2 + \frac{\beta}{3}q_{xx} + \alpha g_1(x+t) + \beta g_2(x+t)$$

According to our seventh assumption, the disturbance $\alpha g_1(x+t) + \beta g_2(x+t)$ that travels to the left vanishes.So, $g_1 = g_2 = 0$ Thus substituting $w = q - \frac{\alpha}{4}q^2 + \frac{\beta}{3}q_{xx}$ in eqn(2.13) or eqn(2.14)and using $w = q$ and $w_t = -w_x$ in the first order terms we obtain

$$q_t + q_x + \frac{3\alpha}{2}qq_x + \frac{\beta}{3}q_{xxx} = 0 \tag{2.19}$$

This is a modified KdV equation due to the term q_x but this term will disappear if we consider $q \to q + C$. But the problem in our transformation is that it is unreal as $C = O(\alpha^{-1})$ will be large and it may violate the boundary conditions. Hence we have the modified KdV equation (2.19) that admits one soliton solutions of the hyperbolic secans form.

2.2 Conservation Laws of KdV Equation

We know KdV equation has infinitely many conservation laws. Here, we discuss about the three basic conservation laws of the equation. For which we considered three basic quantities.

- $Mass = \int_{-\infty}^{\infty} udx$
- $Momentum = \int_{-\infty}^{\infty} u^2dx$
- $Energy = \int_{-\infty}^{\infty}(\frac{1}{2}u_x - u^3)dx$

Now, by using the KdV equation of the form,

$$u_t + (3u^2 + u_{xx})_x = 0 \tag{2.20}$$

For the conservation of mass,

$$\int_{-\infty}^{\infty} udx = constant$$

Thus, it proves conversation of mass. Now, multiplying and integrating eqn(2.10) by $2u$ and integrating, we get

$$\frac{d}{dx}\int_{-\infty}^{\infty} u^2 + 4\int_{-\infty}^{\infty} u_x^3dx + 2\int_{-\infty}^{\infty} uu_{xxx}dx = 0$$

Assuming $u \to 0$ and $u_x \to 0$ as $|x| \to \infty$ the above equation reduced to

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty} u^2 dx &= -4u_x^3|_{-\infty}^{\infty} - 2\int_{-\infty}^{\infty} udu_{xx} \\ &= -2uu_{xx}|_{-\infty}^{\infty} + 2\int_{-\infty}^{\infty} u_x u_{xx} dx \\ &= (u_x^2)_x|_{-\infty}^{\infty} \\ &= 0\end{aligned}$$

$\Rightarrow$

$$\int_{-\infty}^{\infty} u^2 dx = constant$$

Thus, conservation of momentum is proved. Again, multiplying $6u^2$ with eqn(2.10) and integrating we get,

$$2\frac{d}{dt}\int_{-\infty}^{\infty} u^3 dx + 9\int_{-\infty}^{\infty} u_4^4 + 6\int_{-\infty}^{\infty} u^2 u_{xxx} dx = 0$$

Since, we already assumed $u \to 0$ and $u_x \to 0$ as $|x| \to \infty$, we get

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty} u^3 dx &= -6\int_{-\infty}^{\infty} u^2 d_{xx} \\ &= -6uu_{xx}|_{-\infty}^{\infty} + 12\int_{-\infty}^{\infty} uu_x u_{xx} dx \\ &= 6\int_{-\infty}^{\infty} udu_x^2 \\ &= -6\int_{-\infty}^{\infty} u_x^3 dx\end{aligned} \tag{2.21}$$

Now, differentiating eqn(2.10),we get

$$u_{xt} + 6u_x^2 + 6uu_{xx} + u_{xxx} = 0$$

Multiplying $2u_x$ with above equation we get,

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty} u_x^2 dx &= -12\int_{-\infty}^{\infty} u_x^3 dx - 6\int_{-\infty}^{\infty} u du_x^2 - 2\int_{-\infty}^{\infty} u_x du_{xxx} \\ &= -6\int_{-\infty}^{\infty} u_x^3 dx + 2\int_{-\infty}^{\infty} u_{xx}u_{xxx} dx \\ &= -6\int_{-\infty}^{\infty} u_x^3 dx + u_{xx}^2|_{-\infty}^{\infty} \\ &= -6\int_{-\infty}^{\infty} u_x^3\end{aligned} \tag{2.22}$$

From eqn(2.11) and eqn(2.12), we get,

$$\frac{d}{dt}\int_{-\infty}^{\infty}(u_x^3 - 2u^2)dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty}(\frac{1}{2}u_x^3 - u^2)dx = constant$$

Thus, conversation of energy is proved.

2.3 Application Area

With the remarkable discovery of KdV equation and its soliton property opens the gateway to investigate other physical phenomena. KdV equation has approximately describes the evolution of long, one-dimensional waves including: Shallowwater with weakly nonlinear restoring forces, long internal waves in a density-stratified ocean, ion acoustic waves in a plasma, acoustic waves on a crystal lattice, nonlinear evolution of plasma waves, for propagation of transverse waves in a molecular chain, generalised elastic solid, travelling waves, double homoclinic orbit, even now travelling waves of tsunami and red spot of Jupiter can be explained by the KdV equation[25]. KdV plays important role in the development of soliton theory that had lasting impact on today's researches. The theory allows us to study the waves in canals, surface waves near beaches and even under the ocean surface. Soliton solution always interested researchers from the background of oceanography and geophysics. Few of its applications were discussed below.

Internal solitons in the ocean

The internal solitons[25, 26] were seen in the seas in the far east, are very large and dangerous for humans. Even in Andaman sea, these waves were experienced by areas of oil drilling machines. Photos from satellites confirmed that surface waves of vary low amplitude were generated by an internal surface wave resonance mechanism. The presence of such waves supports the theory of kdv internal solitons. Similarly, observations of large amplitude waves were seen in the Sulu sea of southwestern side of Philippines. Many studies shown the presence of oceanic internal solitons in the strait of Messina, Gulf of California etc. These large amplitude waves are weekly non-linear and are of long wavelength that can be described by KdV and the KdV theory indeed gives an excellent explanation of all their features.

Nonlinear Acoustics of Bubbly Liquids

One of the best application of KdV is that its application on the Propagation of non-linear acoustic waves in liquids with gas bubbles[25]. Even for small concentration of gas bubbles they have incredible acoustic properties. The bubble theory of liquids modelled as two co-existing continua and there is a difference between prediction or observation for pure liquid or gas phase. And this prediction or observation of the liquid with gas bubble concentration produces finite number of solitons with no shock-like features under appropriate circumstances.

Magma Flow and Conduit Waves

KdV theory is not confined to only mathematics and physics instead it plays incredible role in several areas of geophysics[25, 26]. Like in conduit flows, buoyant fluid introduced below the layer of greater viscosity, rises through a conduits was created with buoyancy and viscous shear stress which balances the study flow of fluid in a conduit of uniform area. Any variation in the rate of supply generates balanced lumps of conduit waves that propagates upward. And it was thought such diffusion free conduit wave transport magma from Earth's interior to the surface due to which hot spots and volcanic island chains were formed. Though its study is far beyond the reach of the KdV limit but the properties of KdV helps in understanding the experiment like the way they were conducted and the inferences.

Another Phenomenon of geophysics, where KdV plays a role is compaction-driven flow in which buoyant melt is forced through deformable porous crystalline matrix on the Earth's mantle and corresponds to the viscous liquid outside to a conduit. These solitary waves are called magmons[25] but collision between two magmons is not confined to phase shifts only. Magmons are different from conduit waves as they do not transport matters, rather the particles of the fluid experiences a finite displacement in the passage of a magmons whereas in conduit solitary waves, transportation of matter takes place through mantle of the earth with negligible diffusion. It has been assumed that in every 500 million years, the occurrence of single solitary wave would double the magma flux to the Earth's surface which was produced by steady flow in the same conduit.

Jupiter

Till dated, the best application of KdV theory was in understanding the GRS(Great Red Spot)[25] features of South Equatorial Disturbances, the Dark South Tropical Streak, the White Oval and South Tropical Disturbances. The atmospheres of rapidly rotating planetary objects, the linear waves modes are highly dispersive westward-propagating Rossby waves. The long waves travels with highest velocity and are weakly dispersive in nature that leads to a KdV type cubic dispersion. Such wave can preserve for longer period of time, for which the vertical structures should be anticyclonic. This anticyclonic circulation can be observe on the planets like Jupiter, Saturn and Neptune. The vertical structure of the non-linear approximation is featured by KdV. So, by KdV theory we can explain various occurrences that happened in planetary atmospheres.

Plasma physics

KdV play important role in plasma physics also[25]. Plasma waves were seen in hydromagnetics and in ion-acoustic waves in a cold plasma. Ion-acoustic waves gives the first experimental evidence of cylindrical KdV solitons. The cylinder containing plasma was excited by providing bias voltage across the plasma sheath. We can observe soliton like structure from both direction of the axis when input signals are given by the speed of soliton increases in comparing to the speed of ion-acoustic low-frequency speed. Though, integrable study of cylindrical KdV was done but the experimental test of plasma is difficult and are of high cost. The two non-linear weakly waves are not integrable so we need to consider only one way waves for a period such that CKdV gave the diverging waves because of creation of plasma at the inner radius of the cylinder and creating these diverging waves on the surface of water is difficult.

Tsunami

The most recent application of KdV equation is understanding the evaluation and propagation of waves of tsunami[$25-27$]. A tsunami is a very long ocean wave caused by an underwater earthquake, a submarine, volcanic eruption or may be due to land slide. As KdV equation is used to describe shallow water waves of long wavelength and small amplitude. So, it was assumed that the propagation of tsunami waves are along single direction. That can be shown by KdV equation as KdV equation can be elevated and travels faster with amplitude. For future research scientist can show $3D$ tsunami wave can be studied using this.

The periodic wave solution were used by coastal engineers to study sediment movement, erosion of sandy beaches and other costal occurrences. Even in atmospheric science this model helps to study the inertia-gravity waves, vortex interactions and Rossby waves.

2.4 Different Forms of KdV Equation

As we have discussed in chapter 1 about the Korteweg de-Vries(KdV) Equation, its history and its implications too. Here, we gave some of its important forms.

Modified Korteweg de-Vries(MKdV) Equation

The modified korteweg de-vries equation has numerous applications. Like MKdV appears in plasma and even in pulse travelling solution[15]. This problems even aries while evaluating long-time nonlinear wave equation[16]. And the modified Korteweg de-Vries eqaution is given as

$$u_t + 6u^2 u_x + u_{xxx} = 0, x \epsilon R$$

Complex Korteweg de-Vries(CKdV) Equation

In various field of physics and mathematics, complex KdV equation appears. Mostly, in non-linear optics context of solitons and in plasma physics. While studying the solitons through optical fibers, this equation of cKdV was seen. Even in Sasa-Satsuma equation, the propagation of solitons through optical fibers also gives cKdV equation[$22 - 24$]. The general equation is given by

$$w_t + a|w|^m w_x + b w_{xxx} = 0$$

where q represents complex variable and x and t are independent variables. And a and b are real valued constants.

Complex Modified Korteweg de-Vries(CMKdV) Equation

As nature has shown various phenomena relating to partial differential equations like plasma waves, propagation of transverse waves in a molecular chain model[18] and a generalized elastic solid. Because of which researchers deal such problems with complex modified KdV equation. The complex modified Korteweg de-Vries equation is given as

$$\frac{\partial w(x,t)}{\partial t} + \alpha \frac{\partial (|w(x,t)|^2 w(x,t))}{\partial x} + \frac{\partial^3 w(x,t)}{\partial x^3} = 0$$

where w is complex valued function of the spatial coordinate x and the time t, α is a constant parameter.

Coupled Korteweg de-Vries Equation

The coupled KdV equation[$12-14$] are studied due to its significance importance in theoretical physics and other scientific applications. Interest in couple KdV equation arises because of soliton's property after collision. As it is easy to distinguish between two KdV solitons before and after collision but at the time of interaction of the two solitons it creates some confusions. To clarify such confusions coupled KdV equation were introduced and studied. Some examples where coupled KdV were involved are singularity analysis of prolongation technique for developing new coupled KdV equation and spectral problem with three potentials for developing hierarchy of coupled KdV equation. We can see different types of coupled KdV equation. Like,

$$\partial_t u_k + \partial_x[\frac{u_k}{2}\sum_{j=1}^{N} u_j + \partial_x^2 u_k] = 0, k = 1, 2, ..., N$$

For $N > 1$, this system of coupled equations gives a symmetrical multi component KdV equation. Again, the system equations

$$\begin{aligned} r_t &= -r_{xxx} + \frac{7}{4}rr_x + ss_x - \frac{5}{4}(rs)_x \\ s_t &= -s_{xxx} + \frac{5}{4}rr_x + \frac{7}{4}ss_x - 2(rs)_x \end{aligned}$$

and

$$\begin{aligned} r_t &= -r_{xxx} - 3rr_x - 3ww_x \\ s_t &= -s_{xxx} - 3ss_x - 3ww_x \\ w_t &= -w_{xxx} - \frac{3}{2}(rw)_x - \frac{3}{2}(sw)_x \end{aligned}$$

represents the hierarchy coupled KdV equation. And again considering two-layer fluid model, we get systems of equation

$$\begin{aligned} q_{1t} + J\psi_1, q_1 + \beta\psi_{1x} = 0 \\ q_{2t} + J\psi_2, q_2 + \beta\psi_{2x} = 0 \end{aligned}$$

where

$$\begin{aligned} q_1 = \psi_{1xx} + \psi_{1yy} + F(\psi_2 - \psi_1) \\ q_2 = \psi_{2xx} + \psi_{2yy} + F(\psi_1 - \psi_2) \end{aligned}$$

and

$$Ja, b \equiv a_x b_y - b_x a_y$$

This equations help in deriving coupled KdV equation by Multiple scale-approach. KdV was originally derived to explain the shallow water waves in rectangular channels with constant depth. However, by doing some modification and extension in the original equation we can modelled them to explain and help in understanding of the waves like ion-acoustic wave, hydrodynamic waves in plasma, acoustic waves on a crystal lattice and many more.

2.5 Summary of Theoretical review

The most common water waves that we encounter normally are waves at the beach that caused because of action of wind or tides, waves by throwing stones on the pond, by a ship or by the raindrops in river. Such water waves are known as shallow water waves or solitary waves because of their soliton properties. This solitons acts like a particle and shows similar behaviour as a particle. The mathematical model of such waves is represented by Korteweg-de Vries (KdV) equation and we already discussed about KdV equation in previous sections. There are numerous experiments that were conducted and studied by researchers that concluded this property. Now, we will go a little further into the details of theory of this equation, how it may fits in our topic. Thus, we want to know whether differential quadrature and multi-domain in KdV equation gives better results or results similar to previous ones or not.

Some of the previous papers like Saka[17] used cosine expansion based differential quadrature method to find the numerical solution of KdV equation.

And the result shows that CDQM gives more accurate solution with only few discrete points. So, this can be used as alternative numerical method to find accurate result of any differential equation. Korkmaz and Dağ[18]used cosine expansion based differential quadrature method to evaluate the numerical approximation results of CMKdV equation. The results shows high accuracy results. Including that the cost of computation comparing to others is very low. Korkmaz[19]again studied numerical solution of KdV equation by using Lagrange polynomial based differential quadrature(PDQ) and cosine expansion based differential quadrature(CDQ) methods. The numerical approximation results of KdV equation comparing with both the quadrature methods shows that CDQ gave the best results with high accuracy, better memory storage and less computational cost.

Ma and Sun[20] applied a Legendre-Petrov-Garlerkin Chebyshev Collocation method on nonlinear problem like Korteweg-de Vries equation along with multidomain decomposition. Though they did not get the required results but still constructing such method with multidomain technique is very interesting. Pavoni[21] used Chebyshev Collocation algorithm for the numerical approximation of Korteweg-de Vries equation along with single and multidomain decomposition method. The method seems to be very natural. The flexibility nature of multidomain method makes it easier for the differential equation problem to be very accurate near the discrete points where solution can be obtained more precisely.

Thus in this paper we want to use differential quadrature method along with multidomain decomposition technique to study the numerical approximation of Korteweg-de Vries (KdV) equations. And compare its results and nature with results of other numerical techniques.

Chapter 3

DIFFERENTIAL QUADRATURE METHODS

In this chapter, we will be discussing about the different methods of differential quadrature and their recurrence formulas for determining the weighting coefficients.

3.1 Lagrange Based Differential Quadrature

In this method, Lagrange interpolating polynomials are used as base functions. Let us consider a set of Lagrange interpolating polynomial as[10, 29]

$$r_k(x) = \frac{M(x)}{(x - x_k)M^{(1)}(x_k)}, \quad k = 0, 1, 2, ...N \tag{3.1}$$

where

$$M(x) = (x - x_0)(x - x_1)...(x - x_N)$$

and

$$M^{(1)}(x_k) = \prod_{l=0, l \neq k}^{N} (x_l - x_k)$$

constitutes an $N + 1$ dimensional vector space and used as the test function for determining the weighting coefficients. After doing some calculations and using differential quadrature approximation given in eqn(1.2), the weighting coefficients of the first order derivatives are given by

$$a_{ij} = \frac{M^{(1)}(x_i)}{(x_i - x_j).M^{(1)}(x_j)} \;\; , i \neq j, \;\; i, j = 1, 2, 3, ...N \tag{3.2}$$

For determining the diagonal weighting coefficients, the test function $g_k(x) = 1$ is chosen from the set of test functions $g_k(x) = x^k, k = 0, 1, ..., N$. So the diagonal weighting coefficients are given by

$$\sum_{j=1}^{N} a_{ij} = 0 \;\; or \;\; a_{ii} = -\sum_{j=1, j \neq i}^{N} a_{ij} \;\; , i = j \tag{3.3}$$

Similarly, the weighting coefficients for second order derivatives are given by

$$b_{ij} = 2a_{ij}[a_{ij} - \frac{1}{(x_i - x_j)}] \;\; , i \neq j \tag{3.4}$$

and

$$\sum_{j=1}^{N} b_{ij} = 0 \;\; or \;\; b_{ii} = -\sum_{j=1, j \neq i}^{N} b_{ij} \;\; , i = j \tag{3.5}$$

And the Recurrence Formulaes for weighting coefficients of higher order derivatives are given by

$$w_{ij} = m[w_{ii}^{m-1} a_{ij} - \frac{w_{ij}^{(m-1)}}{(x_i - x_j)}] \ , i \neq j, \ \ m = 2, 3, ...N-1 \tag{3.6}$$

and

$$\sum_{j=1}^{N} w_{ij}^{(m)} = 0 \ \ or \ \ w_{ii}^{(m)} = - \sum_{j=1, j \neq i}^{N} w_{ij}^{(m)} \ \ , i = j \tag{3.7}$$

3.2 Fourier Expansion Based Differential Quadrature

Here fourier series expansion is used for the approximation of $f(x)$, which is of the form[48]

$$f(x) = c_0 + \sum_{k=1}^{N/2} (c_k coskx + d_k sinkx)$$

Now, $f(x)$ constitutes a $(N+1)$ dimensional vector space. In linear vector space there are two typical sets of base vectors, which are used in the formulation of FDQ

$$1, cosx, sinx, cos2x, sin2x, ..., cos(Nx/2), sin(Nx/2)$$

and

$$s_k(x) = \frac{S(x)}{q(x_k) sin \frac{x - x_k}{2}}, \ \ k == 0, 1, ..., N$$

where

$$S(x) = \prod_{k=0}^{N} sin \frac{x - x_k}{2}$$

$$q(x_i) = \prod_{k=0,k\neq i}^{N} sin\frac{x_i - x_k}{2}$$

So after some calculations, the weighting coefficients of first order derivative are determined by

$$a_{ij} = \frac{q(x_i)}{2sin\frac{x_i - x_j}{2} q(x_j)}, \;\; j \neq i \tag{3.8}$$

and by using the base vector 1, we get

$$\sum_{j=1}^{N} a_{ij} = 0 \;\; or \;\; a_{ii} = -\sum_{j=1,j\neq i}^{N} a_{ij} \;\; , i = j \tag{3.9}$$

Similarly, the weighting coefficients for second order derivative are given by

$$b_{ij} = a_{ij}[2a_{ij} - cot\frac{x_i - x_j}{2}], \;\; j \neq i \tag{3.10}$$

and

$$\sum_{j=1}^{N} b_{ij} = 0 \;\; or \;\; b_{ii} = -\sum_{j=1,j\neq i}^{N} b_{ij} \;\; , i = j \tag{3.11}$$

3.3 Cosine Expansion Based Differential Quadrature

Here fourier series expansion is used for the approximation of an even function $f(x)$, which is of the form$[10, 17]$

$$f(x) = d_0 + \sum_{k=1}^{N} d_k coskx$$

Now, in a linear vector space of $N+1$ dimension, two typical sets of base vectors are used to determine the CDQ

$$C_k(x) = cos(kx), k = 0, 1, 2, ..., N$$

and

$$C_k(x) = \frac{C(x)}{P(x_k)(cosx - cosx_k)}, \;\; k = 0, 1, 2, ..., N$$

where

$$C(x) = \prod_{k=0}^{N}(cosx - cosx_k)$$

$$P(x_i) = \prod_{k=0,k\neq i}^{N}(cosx_i - cosx_k)$$

after doing some calculations and using eqn(1.2), we get the first order weighting coefficients of CDQ as

$$a_{ij} = \frac{-P(x_i)sin(x_i)}{(cosx_i - cosx_j)P(x_j)}, \;\; j \neq i \tag{3.12}$$

and

$$\sum_{j=1}^{N} a_{ij} = 0 \;\; or \;\; a_{ii} = -\sum_{j=1,j\neq i}^{N} a_{ij}, \;\; i = j \tag{3.13}$$

Similarly, weighting coefficients for second order derivative are given by

$$b_{ij} = a_{ij}(\frac{2sinx_i}{(cosx_i - cosx_j)}) + 2a_{ii} + cotx_i, \;\; j \neq i \tag{3.14}$$

and

$$\sum_{j=1}^{N} b_{ij} = 0 \;\; or \;\; b_{ii} = -\sum_{j=1,j\neq i}^{N} b_{ij}, \;\; i = j \tag{3.15}$$

And for the higher order derivative weighting coefficients are given by

$$w_{ij}^{(3)} = 3w_{ij}^{(1)}(w_{ii}^{(2)} - \frac{1}{3} + w_{ii}^{(1)}cotx_i + \frac{cosx_i}{cosx_i - cosx_j}) + \frac{3sinx_i.w_{ij}^{(2)}}{cosx_j - cosx_j}, \quad i \neq j \quad (3.16)$$

3.4 Sine Expansion Based Differential Quadrature

Here fourier series expansion is used for the approximation of an odd function $f(x)$, which is of the form[10]

$$f(x) = \sum_{k=1}^{N} d_k coskx$$

Now, in a linear vector space of $N+1$ dimension, two typical sets of base vectors are used to determine the SDQ

$$S_k(x) = sin(kx), k =, 1, 2, ..., N$$

and

$$S_k(x) = \frac{sinxC_k(x)}{sinx_k}$$

where

$$C_k(x) = \frac{C(x)}{P(x_i)(sinx - sinx_i)}$$

where

$$C(x) = \prod_{k=1}^{N}(sinx - sinx_i)$$

$$P(x_i) = \prod_{k=1, k\neq i}^{N}(sinx_i - sinx_k)$$

after doing some calculations and using eqn(1.2), we get the first order weighting coefficients of SDQ as

$$a_{ij} = \frac{-P(x_i)sin^2x_i}{(cosx_i - cosx_j)sinx_j.P(x_j)}, \ \ j \neq i \tag{3.17}$$

and

$$a_{ii} = c_i^{(1)}(x_j) + cotx_i, \tag{3.18}$$

Similarly the second order weighting coefficients are given by

$$b_{ij} = a_{ij}(\frac{2sinx_i}{(cosx_i - cosx_j)} + 2a_{ii} + cotx_i, \ \ j \neq i \tag{3.19}$$

and

$$b_{ii} = C_i^{(2)}(x_i) + 2cotx_i.C_i^{(1)}(x_i) \ \ i = j \tag{3.20}$$

And for the higher order derivative weighting coefficients are given by

$$w_{ij}^{(3)} = 3a_{ij}^{(1)}(b_{ii} - \frac{1}{3} + a_{ii}cotx_i + \frac{cosx_i}{cosx_i - cosx_j}) + \frac{3sinx_i.b_{ij}^{(2)}}{cosx_j - cosx_j}, \ \ i \neq j \tag{3.21}$$

Here, we can see that eqn(3.12) and eqn(3.17) are slightly different but eqns(3.14) and (3.16) are exactly same as eqn(3.19) and (3.21). Thus, the second and higher order derivatives of both the cosine and sine expansion based differential quadrature methods uses the same formulae for the computation of weighting coefficients for $i \neq j$ and the diagonal weighting coefficients ($i = j$) are computed by different formulations.

3.5 Quintic B-Spline

In this section, we will be discussing about the basis function i.e Quintic B-Spline to determine the weighting coefficients for the differential quadrature. Now, consider $Q_m(x)$ be the quintic B-splines with knots x_i where uniformly distributed $\mathbb{N}$ grid points are chosen as $a = x_1 < x_2 < \cdots\cdots < x_N = b$ on the real axis. Then, the splines $Q_{-1}, Q_0, Q_1, \cdots\cdot, Q_{N+2}$ form the basis functions defined over $[a, b]$. The quintic B-spline is given by the following recurrence relationship[28, 29]:

$$Q_m(x) = \frac{1}{h^5}\begin{cases} (x - x_{m-3})^5 & , [x_{m-3}, x_{m-2}) \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 & , [x_{m-2}, x_{m-1}) \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 + 15(x - x_{m-1})^5 & , [x_{m-1}, x_m) \\ (x_{m+3} - x)^5 - 6(x_{m+2} - x)^5 + 15(x_{m+1} - x)^5 & , [x_m, x_{m+1}) \\ (x_{m+3} - x)^5 - 6(x_{m+2} - x)^5 & , [x_{m+1}, x_{m+2}) \\ (x_{m+3} - x)^5 & , [x_{m+2}, x_{m+3}) \\ 0 & , \text{otherwise} \end{cases} \tag{3.22}$$

where $h = x_i - x_{i-1}$ for all i. Each quintic B-spline have six elements, in order to cover each element with six quintic B-splines. The value of $Q_m(x)$ and its derivatives are given in the Table 3.1. We reduces the stated problem to the derivative of the fifth order B-spline with unknown coefficients. And the weighting coefficients are determined by five banded Thomas algorithm for penta-diagonal systems. Now using quintic B-spline as test function in differential quadrature methods eqn 1.2, we get

Table 3.1: $Q_m(x)$and derivatives at the grid points

x	x_{m-3}	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}	x_{m+3}
Q_m	0	1	26	66	26	1	0
Q'_m	0	$\frac{5}{h}$	$\frac{50}{h}$	0	$\frac{-50}{h}$	$\frac{-5}{h}$	0
Q''_m	0	$\frac{20}{h^2}$	$\frac{40}{h^2}$	$\frac{-120}{h^2}$	$\frac{40}{h^2}$	$\frac{20}{h^2}$	0
Q'''_m	0	$\frac{60}{h^3}$	$\frac{-120}{h^3}$	0	$\frac{120}{h^3}$	$\frac{60}{h^3}$	0
Q''''_m	0	$\frac{120}{h^4}$	$\frac{-480}{h^4}$	$\frac{720}{h^4}$	$\frac{480}{h^4}$	$\frac{120}{h^4}$	0

the following equation i.e,

$$\frac{\partial^{(p)} Q_m(x_i)}{\partial x^{(p)}} = \sum_{j=m-2}^{m+2} w_{ij}^{(p)} Q_m(x_j), i = 1, 2, 3, \cdots, N, m = -1, 0, 1, \cdots, N+1, N+2 \tag{3.23}$$

Now, arbitrary choice of i leads to a system algebraic equations,

$$\begin{bmatrix} Q_{-2,-4} & Q_{-2,-3} & Q_{-2,-2} & Q_{-2,-1} & Q_{-2,0} & \\ & Q_{-1,-3} & Q_{-1,-2} & Q_{-1,-1} & Q_{-1,0} & Q_{-1,1} \\ & \ddots & \ddots & \ddots & \ddots & \\ & Q_{N+1,N-1} & Q_{N+1,N} & Q_{N+1,N+1} & Q_{N+1,N+2} & Q_{N+1,N+3} \\ & Q_{N+2,N} & Q_{N+2,N+1} & Q_{N+2,N+2} & Q_{N+2,N+3} & Q_{N+2,N+4} \end{bmatrix} \begin{bmatrix} w_{i,-4}^{(p)} \\ w_{i,-3}^{(p)} \\ \vdots \\ w_{i,N+3}^{p} \\ w_{i,N+4}^{p} \end{bmatrix} = \mathbf{\Psi} \tag{3.24}$$

where $Q_{i,j}$ means $Q_i(x_j)$ and $\mathbf{\Psi} = \left[\frac{\partial^{(p)} Q_{-2}(x_{(i)})}{\partial x^{(p)}} , \frac{\partial^{(p)} Q_{-1}(x_i)}{\partial x^{(p)}} , \cdots\cdots , \frac{\partial^{(p)} Q_{N+1}(x_i)}{\partial x^{(p)}}, \frac{\partial^{(p)} Q_{N+2}(x_i)}{\partial x^{(p)}} \right]^T$.
The weighting coefficients $w_{i,j}^{(p)}$ can be determined by using eqn(3.24) . The eqn(3.24) has $N+5$ equations and $N+9$ unknowns. In order to solve eqn(3.24) we required additional four equations

$$\frac{\partial^{(p+1)} Q_{-2}(x_i)}{\partial x^{(p+1)}} = \sum_{j=-4}^{0} w_{i,j}^{(p)} Q'_{-2}(x_j)$$

$$\frac{\partial^{(p+1)} Q_{-1}(x_i)}{\partial x^{(p+1)}} = \sum_{j=-3}^{1} w_{i,j}^{(p)} Q'_{-1}(x_j)$$

$$\frac{\partial^{(p+1)} Q_{N+1}(x_i)}{\partial x^{(p+1)}} = \sum_{j=N-1}^{N+3} w_{i,j}^{(p)} Q'_{N+1}(x_j)$$

$$\frac{\partial^{(p+1)} Q_{N+2}(x_i)}{\partial x^{(p+1)}} = \sum_{j=N}^{N+4} w_{i,j}^{(p)} Q'_{N+2}(x_j)$$

By using the functional values of quintic B-spline at the grid points and eliminating $w_{i,-4}^{(p)}$, $w_{i,-3}^{(p)}$, $w_{i,N+3}^{(p)}$, and $w_{i,N+4}^{(p)}$ from the system. We obtained a system of algebraic equation having five-banded coefficient matrix of the form

$$\mathbf{M}\mathbf{w} = \mathbf{\Psi} \tag{3.25}$$

where

$$\mathbf{M} = \begin{bmatrix} 37 & 82 & 21 & & & & & & & \\ 8 & 33 & 18 & 1 & & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & & \\ & & \ddots & \ddots & \ddots & \ddots & & & & \\ & & & 1 & 26 & 66 & 26 & 1 & & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 1 & 18 & 33 & 8 \\ & & & & & & 21 & 82 & 37 \end{bmatrix} \quad and \quad \mathbf{w} = \begin{bmatrix} w_{i,-2}^{(p)} \\ w_{i,-1}^{(p)} \\ \vdots \\ w_{i,j-2}^{(p)} \\ w_{i,j-1}^{(p)} \\ w_{i,j}^{(p)} \\ w_{i,j+1}^{(p)} \\ w_{i,j+2}^{(p)} \\ \vdots \\ w_{i,N+1}^{(p)} \\ w_{i,N+2}^{(p)} \end{bmatrix}$$

The non-zero entries of the load vector $\mathbf{\Psi}$ are given by

$$\begin{aligned}
\mathbf{\Psi}_{-2} &= \frac{1}{30}(-5Q_{-2}^{(p)}(x_i) - hQ_{-2}^{(p+1)}(x_i) + 40Q_{-1}^{(p)}(x_i) + 8hQ_{-1}^{(n+1)}(x_i)) \\
\mathbf{\Psi}_{-1} &= \frac{1}{10}(5Q_{-1}^{(p)}(x_i) - hQ_{-1}^{(p+1)}(x_i)) \\
\mathbf{\Psi}_{m-2} &= Q_{m-2}^{(p)} \\
\mathbf{\Psi}_{m-1} &= Q_{m-1}^{(p)} \\
\mathbf{\Psi}_{m} &= Q_{m}^{(p)} \\
\mathbf{\Psi}_{m+1} &= Q_{m+1}^{(p)} \\
\mathbf{\Psi}_{m+2} &= Q_{m+2}^{(p)} \\
\mathbf{\Psi}_{N+1} &= \frac{1}{10}(5Q_{N+1}^{(p)}(x_i) - hQ_{N+1}^{(p+1)}(x_i)) \\
\mathbf{\Psi}_{N+2} &= -\frac{1}{30}(5Q_{N+1}^{(p)}(x_i) + hQ_{N+2}^{p+1}(x_i) - 40Q_{N+1}^{(p)}(x_i) + 8hQ_{N+1}^{(p+1)}(x_i))
\end{aligned} \tag{3.26}$$

The above equations can be solved by using Thomas algorithm at each grid points x_i, $i = 1, 2 \cdots N$ and we can get all the weighting coefficients $w_{i,j}^{(p)}$, for $i, j = 1, 2, \cdots N$, $p = 1, 2, \cdots N - 1$. Here, we calculated the weighting coefficient of first order i.e $w_{i,j}^{(1)}$ at each grid points x_i, using eqn(3.25) and the above load vectors. As KdV equation involved third order derivative. So, we calculated the weighting coefficients of third order derivative i.e, $w_{i,j}^{(3)}$ at the grid points x_i using the eqn(3.25) and eqn(3.26).

3.6 Multidomain PDQ Method

As we know from the above section differential quadrature methods carried problems that involves simple domains. So, it requires computational domains to be regular so that the the boundary could be a mesh line. And if the boundary is curved, then applying differential quadrature method will be difficult. So, in order to overcome this difficulties two approaches were made[10, 48],

- Multi-Domain Approach:- In this approach, the whole domain is decomposed into several subdomains and differential quadrature discretization method is applied in each subdomain.

- Coordinate Transformation Approach:- In this approach, the irregular domain in the cartesian coordinate is transformed into the regular domain in the curvilinear coordinate system and the problems in the cartesian system are transformed into appropriate forms in the curvilinear system. Then the numerical computations are performed on the curvilinear coordinate system.

We will discuss a little further about multidomain differential quadratue method. Shu(1991)was the first to proposed the Multi-domain differential quadrature method[10] to simulate incompressible flows past a backward facing step and a square step. Then in 1992 Shu and Richards applied it to stimulation of driven cavity flows on a multi instruction , multi data-stream computer. In 1998 Shu, Chew and liu used it in the stimulation of flows in the Czochralski crystal growth, Shu and Chew in waveguide analysis with rectangular boundaries, Zhong and He in computation of Poisson and Laplace equations and in structural and vibration analysis.

Now in Multi-Domain DQ Method the computational domain of a problem is represented by Ω and the boundary by Γ. Here, the multi domain method decomposes the domain Ω into several subdomains and each subdomain generated a local mesh and differential quadrature method is applied in same way as in single domain. Now through interface of subdomains the information were exchanged. Now there are two type of interfaces : patched and overlapped.

- Patched Interface $\rightarrow$ In the interface as in Figure3.1, Γ_{ij} is the interface between two subdomains Ω_i and Ω_j. Here, the main equation is not applied along the interface rather continuity condition is enforced. Main thing to let the function and its normal derivative to be continuous at the interface. Mathematically, this continuity condition can be given as

  $$f(x_N^i) = f(x_I^j) on \Gamma_{ij}$$

  $$f_n(x_N^i) = f_n(x_I^j) on \Gamma_{ij}$$

  where $f(x_N^i)$ and $f(x_I^j)$ represent the value of the function at the interface of the i-subdomain and j-subdomain respectively and whereas, $f_n(x_N^i)$ and $f_n(x_I^j)$

the value of the first order derivatives of w.r.t n at the interface.

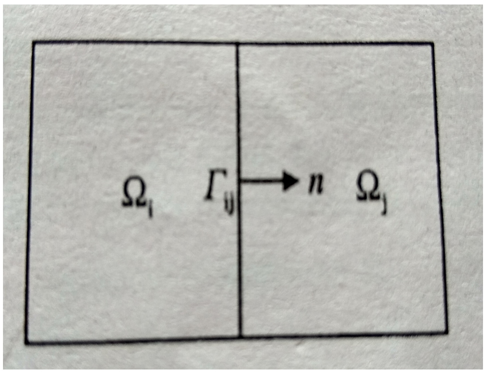

Figure 3.1: Topology of a patched interface

- Overlapped Interface $\rightarrow$ In this interface as in Figure3.2, subdomain ABCD overlapped subdomain EFGH. The right boundary of the subdomain Ω_i,BC is the interior of subdomain Ω_j and the left boundary of the subdomain Ω_j, EH is the interior of subdomain Ω_i.

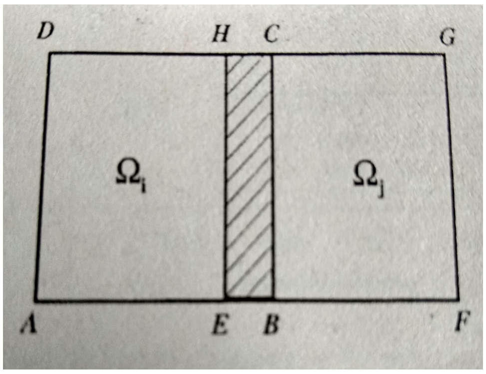

Figure 3.2: Topology of an overlapped interface

Here, we considered the Multidomain pseudo-spectral method[29, 30]. Suppose $P_N I$ be the space for all algebraic polynomials up to degree N on the $[-1, 1]$. The multidomain polynomial based differential quadrature(MD PDQ) method for the equation

$\theta u = F$ where θ is the differential operator. Now, we need to find $u \epsilon P_N I$ for which

$$(\theta u - F)(\eta_j)\eta = 0, \eta_j = -cos(\frac{j\Pi}{N}), j = 0, 1, 2, ..., N \tag{3.27}$$

satisfying the boundary conditions. We subdivided the whole solution domain $I_x = [a, b]$ into M uniform subintervals i.e $I^n = [x_L^n, x_R^n]$, $I_x = [a, b] = \bigcup_n^M I_n$. Without the loss of generality, we consider the length of each subinterval as $l = \frac{b-a}{M+(1-M)(1-cos(\Pi/N))/2}$. In this domain decomposition method, we overlap intervals I^n i.e the first two quadrature points of I^{n+1} coincide with last two quadrature points of I^n. And in each subdomain I^n, we got $\theta u^n = F$ where u^n is the solution over the nth interval.

In order to use the weighting coefficients of differential quadrature on any arbitrary interval $I^n = [x_L^n, x_R^n]$ we use the mapping $\eta : [x_L^n, x_R^n] \rightarrow [-1, 1]$ defined as $\eta(x) = \frac{2}{x_R^n - x_L^n}x - \frac{x_R^n + x_L^n}{x_R^n - x_L^n}$ that maps coordinate x onto η. So, for the derivatives, we have

$$\frac{\partial^p u^n(x)}{\partial x^p} = (\frac{2}{x_R^n - x_L^n})^p \frac{\partial^p u^n(\eta)}{\partial \eta^p}$$

So, the weighting coefficient of the pth order derivative is given as

$$W^p_{[0;N,0;N]} = (\frac{2}{l})^p (w_{ij}^{(p)})_{(N+1)(N+1)} \tag{3.28}$$

Now, for global satisfaction of the equations. For this we did element-wise construction basing on the summation of the local element matrices to form their global representations. Now by interpolating the function u to any arbitrary position in the interval $I_x = [a, b]$ with the new grid points given by

$$x_j = x'_0, \cdots, x'_{N-1} = x_0^2, x'_N = x_1^2, \cdots, x_{N-1}^n = x_0^{n+1}, \cdots, x_N^M$$

where $x_j^n = \frac{1}{2}\eta(x_j^{n-1}) + \frac{l}{2} + x_{n-1}^{N-1}$. Let us assume that $u^n = u^{n+1}$ on the boundary of the overlapping domain, the functional value of u at the grid points can be given by

$$\tilde{\psi} = [\tilde{u}_0^1, \cdots, \tilde{u}_{N-1}^1, \tilde{u}_1^2, \cdots, \tilde{u}_{N-1}^n, \cdots, \tilde{u}_{N-1}^M, \tilde{u}_{N-1}^M]^T$$

where $\tilde{u}_j^n = u(x_j^n)$. Thus the functional values of the derivatives u at the grid points x_j is given by $\tilde{\psi}^{(p)} = \tilde{W}_{[0;k,0;k]}^{(p)}\tilde{\psi}$ where

$$\tilde{W}_{[0;k,0;k]}^{(p)} = \begin{pmatrix} \tilde{w}_{[0,0]}^{(1)} & \cdot\cdot & \tilde{w}_{[0,N]}^{(1)} & & & & & \\ & \ddots & & & & & & \\ \tilde{w}_{[N-1,0]}^{(1)} & \cdot\cdot & \tilde{w}_{[N-1,N]}^{(1)} & & & & & \\ & & \ddots & & & & & \\ & & \tilde{w}_{[1,0]}^{(n)} & \cdot\cdot & \tilde{w}_{[1,N]}^{(n)} & & & \\ & & & \ddots & & & & \\ & & \tilde{w}_{[N-1,0]}^{(n)} & \cdot\cdot & \tilde{w}_{[N-1,N]}^{(n)} & & & \\ & & & & \ddots & & & \\ & & & & \tilde{w}_{[1,0]}^{(M)} & \cdot\cdot & \tilde{w}_{[1,N]}^{(M)} \\ & & & & & \ddots & \\ & & & & \tilde{w}_{[N-1,0]}^{(M)} & \cdot\cdot & \tilde{w}_{[N-1,N]}^{(M)} \end{pmatrix} \quad (3.29)$$

where $[\tilde{\psi}]_{ij}^{(n)} = [W^{(p)}]_{ij}$ and the remaining components of the matrix $\tilde{W}_{[0;k,0;k]}^{(p)}$ with $k = M(N-1)+1$ are zero.

Chapter 4

NUMERICAL EXPERIMENTS

4.1 Implementation of MD-PDQ to KdV equation

In this section, we consider KdV equation of the form[31]

$$u_t + \xi u u_x + \mu u_{xxx} = 0 \tag{4.1}$$

as our governing equation which represents one-dimensional nonlinear KdV equation of shallow water waves. Where $u = u(x,t)$, ξ and μ are the positive parameters. We discritize eqn(4.1) by forward finite difference and Crank-Nicolson,

$$\frac{u^{n+1} - u^n}{\triangle t} + \xi \frac{(uu_x)^{n+1} + (uu_x)^n}{2} + \mu \frac{(u_{xxx})^{n+1} + (u_{xxx})^n}{2} = 0 \tag{4.2}$$

Now by using Rubin and Graves linearization technique[47], we linearize the non-linear terms so we obtain

$$\frac{u^{n+1}-u^n}{\triangle t}+\xi\frac{u^n u_x^{n+1}+u_x^n u^{n+1}}{2}+\mu\frac{(u_{xxx})^{n+1}+u_{xxx}^n}{2}=0$$

$$\Rightarrow 2u^{n+1}+\xi\triangle t u_x^n u^{n+1}+\xi\triangle t u^n u_x^{n+1}+\mu\triangle t u_{xxx}^{n+1}=2u^n-\mu\triangle t u_{xxx}^n \tag{4.3}$$

Let us define some terms to use in eqn(4.3) as

$$A_i^n=\sum_{j=0}^{N}\tilde{w}_{ij}^{(1)}u_j^n,$$
$$B_i^n=\sum_{j=0}^{N}\tilde{w}_{ij}^{(3)}u_j^n$$

where A_i^n and B_i^n are the first order and third order derivative approximations of function u at the nth degree on x_i $(i=0,1,2,....,N)$ points respectively. Now substituting this terms in eqn(4.3) we get,

$$(2+\xi\triangle t A_i^n+\xi\triangle t\tilde{w}_{ii}^{(1)}u_i^n+\mu\triangle t\tilde{w}_{ii}^{(3)})u_i^{n+1}$$
$$+\sum_{j=0,j\neq i}^{N}(\xi\triangle t\tilde{w}_{ij}^{(1)}u_i^n+\mu\triangle t\tilde{w}_{ij}^{(3)})u_j^{n+1}=2u^n-\mu\triangle t C_i^n, i=0,1,....,N \tag{4.4}$$

4.2 Implementation of DQM to KdV equation

In this section, we consider the KdV equation (4.1) and discritize it by using differential quadrature approximation(1.2). Now applying the boundary conditions

$u(a,t) = g_1(t), u(b,t) = g_2(t), t\epsilon(0,T]$ and initial condition,

$$u(x,0) = f(x), a \ll x \ll b$$

we obtain ordinary differential equation of the form

$$\frac{du(x_i)}{dt} = -\xi u(x_i,t)\sum_{j=2}^{N-1} w_{ij}^{(1)} u(x_j,t) - \mu \sum_{j=2}^{N-1} w_{ij}^{(3)} u(x_j,t) + B(u), i = 2,3,\cdots, N-1 \tag{4.5}$$

where$B(u) = -\xi u(x_i,t)[w_{i1}^{(1)} g_1(t) + w_{iN}^{(1)} g_2(t)] - \mu[w_{i1}^{(3)} g_1(t) + w_{iN}^{(3)} g_2(t)]$.
The weighting coefficients $w_{ij}^{(1)}$ and $w_{ij}^{(3)}$ are determined by (3.28). Eqn(4.5) is solved by using RK4 method, since it has advantages of high accuracy, stability and low memory storage. The accuracy of the method is measured by using L_∞ error norm,

$$L_\infty = |U^{exact} - U^{num}|_\infty = max_j |U_j^{exact} - U_j^{num}|$$

As we know there are infinite numbers of conversation laws[32] for KdV equation. And these conservation need to be remain constants during the propagation to show the efficiency of the numerical scheme. So, we have considered the first three conversation quantities,

$$\begin{aligned} C_1 &= \int_b^a U dx \\ C_2 &= \int_b^a U^2 dx \\ C_3 &= \int_b^a (U^3 - \frac{3\mu}{\epsilon} U_x^2) dx \end{aligned}$$

4.3 Result and Discussion

4.3.1 Single Soliton

The soliton solution of KDV equation is of the form[17]

$$U(x,t) = 3csech^2(Ax - Bt + F)$$

The above equation represents a single soliton with velocity ϵc , $A = \frac{1}{2}\sqrt{\frac{\epsilon c}{\mu}}$ and $B = \epsilon cA$. The exact solution at $t = 0$ is taken as initial condition and $U(0,t) = 0$ and $U(2,t) = 0$ as boundary conditions. Considering the parameters $\epsilon = 1$, $\mu = 4.84 \times 10^{-4}$, $c = 0.3$, $F = -6$, we applied our present method MD-PDQ to the given equation and the results thus obtained are tabulated in Table 4.1. In Table.4.1, we presented L_∞ error norm at different times up to $t = 2$. Evolution of single soliton at different time were also shown in Figure4.1. Error norm of the single soliton at $t = 2$ is depicted in Figure4.2. The maximum error norm is found to be very small and all the three invariants are conserved very well. To show the efficiency of our present MD-PDQ method, we used QBDQM and CDQM with the same parameters and the numerical results are tabulated in Table 4.2 and Table 4.3 respectively. Comparison of error with other numerical techniques is shown in Table 4.4.

4.3.2 Interaction of two solitons

The interaction of two solitons solution of KdV equation is given by[33]

$$U(x,t) = 12\frac{3 + 4cosh(2x - 8t) + cosh(4x - 64t)}{[3cosh(x - 28t) + cosh(3x - 36t)]^2}$$

The exact solution at $t = 0$ is taken as initial condition. Considering the parameters $\epsilon = 1$, $\mu = 1$. The computational domain is $[-15, 15] \times [-0.3, 0.3]$. We applied QBDQM and CDQ methods to the given equation and the results thus obtained are tabulated in Table 4.5 and Table 4.6 respectively. In both the tables we presented L_∞ and invariants at different times up to $t = 0.3$. Interaction of two solitons at different times i.e $t = -0.3$, -0.1, 0.1 and 0.3 were also shown in Figure4.3. The maximum error norm is found to be very small and all the three invariants are conserved very well. To show the efficiency of QBDQM and CDQM, maximum error is compared with earlier numerical methods in Table 4.7.

For KdV equation, the accuracy result of MD-PDQ is less because of the space domain $[0, 2]$ as it is very small. Therefore, partitioning of the domain into subdomains leads to large numerical value of the weighting coefficients because of the factors $\alpha = 2/l$ where $l = \frac{b-a}{M+(1-M)(1-cos(\Pi/N))/2}$. Though it gave better results in comparing to earlier methods as in Table 4.4.

Chapter 5

COMPLEX MODIFIED KORTEWEG-de VRIES EQUATION

In this chapter, we will be discussing about the complex modified KdV(CMKdV) equation and applying our present method; MD-PDQ method on the equation to approximate its numerical solution. Now, CMKdV has many applications in Science like it prposed the model for evolution of nonlinear plasma waves[42], describes the propagation of transverse waves in a molecular chain model[43] and in generalized elastic solid[44, 45].

The complex modified Korteweg-de Vries equation is a nonlinear partial differential equation which is of the form ,

$$u_t + u_{xxx} + \alpha(|u|^2 u)_x = 0 \tag{5.1}$$

where u is a complex valued function of the spatial coordinate x and the time t, α is real parameter and subscripts t and x denote differential w.r.t time and space respectively. We decompose u into real and imaginary parts,

$$u(x,t) = r(x,t) + is(x,t) \tag{5.2}$$

where $i = \sqrt{-1}$. Substituting eqn(5.2) in eqn(5.1) yields a couple of modified Korteweg-de Vries(MKdV) equations

$$r_t + r_{xxx} + \alpha[(r^2 + s^2)r]_x = 0 \tag{5.3}$$

$$s_t + s_{xxx} + \alpha[(r^2 + s^2)s]_x = 0 \tag{5.4}$$

Now rearranging the above two equations we get,

$$r_t = -r_{xxx} - \alpha[2rss_x + (3r^2 + s^2)r_x], \tag{5.5}$$

$$s_t = -s_{xxx} - \alpha[2rsr_x + (3s^2 + r^2)s_x] \tag{5.6}$$

where $r(x,t)$ and $s(x,t)$ are real functions.

5.1 Implementation of MD-PDQ to KdV equation

In this section, we discitize the equations(5.5) by using forward finite difference and Crank-Nicolson,

$$\frac{r^{n+1}-r^n}{\triangle t}+\alpha[3\frac{(r^2r_x)^{n+1}+(r^2r_x)^n}{2}+\frac{(s^2r_x)^{n+1}+(s^2r_x)^n}{2} + 2\frac{(rss_x)^{n+1}+(rss_x)^n}{2}]+\frac{(r_{xxx})^{n+1}+(r_{xxx})^n}{2}=0 \quad (5.7)$$

Now by using Rubin and Graves linearization technique[47], we linearize the non-linear terms thus we obtained

$$2r^{n+1}+\triangle t[r_{3x}^{n+1}+3\alpha((r^2)^n r_x^{n+1}+2r^n r_x^n r_x^{n+1})+\alpha((s^2)^n r_x^{n+1}+2s^n r_x^{n+1}+2s^n r_x^n s^{n+1}) + 2\alpha(r^{n+1}s^n s_x^n+r^n s^{n+1}s_x^n+r^n s^n s_x^{n+1})]=2r^n+\triangle t[-r_{3x}^n+3\alpha(r^2)^n r_x^n + \alpha(s^2)^n r_x^n+2\alpha r^n s^n s_x^n] \quad (5.8)$$

Let us define some terms to use in eqn(5.8)

$$A_i^n=\sum_{j=0}^{N}\tilde{w}_{ij}^{(1)}r_j^n,$$

$$B_i^n=\sum_{j=0}^{N}\tilde{w}_{ij}^{(3)}r_j^n,$$

$$C_i^n=\sum_{j=0}^{N}\tilde{w}_{ij}^{(1)}s_j^n,$$

$$D_i^n=\sum_{j=0}^{N}\tilde{w}_{ij}^{(3)}s_j^n$$

where A_i^n and B_i^n are the first order and the third order derivatives approximation of real function $r(x,t)$ at the nth degree on x_i $(i = 0, 1, 2, ..., N)$ points respectively. Similarly, C_i^n and D_i^n are the first order and the third order derivatives approximation of function $s(x,t)$ at the nth degree on x_i $(i = 0, 1, 2, ..., N)$ points respectively. Now substituting this terms in eqn(5.8) we get,

$$\begin{aligned}[2 + \triangle t(\tilde{w}_{ii}^{(3)} + \alpha(3(r_i^n)^2\tilde{w}_{ii}^{(1)} + 6r_i^n A_i^n + (s_i^n)^2\tilde{w}_{ii}^{(1)} \\ + 2s_i^n C_i^n))]r_i^{n+1} + \sum_{j=0, j\neq i}^{N} \triangle t(\tilde{w}_{ij}^{(3)} + \alpha(3(r_i^n)^2\tilde{w}_{ij}^{(1)} \\ + (s_i^n)^2\tilde{w}_{ij}^{(1)}))r_j + [2\alpha\triangle t(s_i^n A_i^n + r_i^n C_i^n + r_i^n s_i^n \\ \tilde{w}_{ii}^{(1)})]s_i^{n+1} + [\sum_{j=0, j\neq i}^{N} (2\alpha\triangle t r_i^n s_i^n \tilde{w}_{ij}^{(1)})s_j^{n+1}] = f_i^n \quad (5.9)\end{aligned}$$

where $f_i = 2r_i^n + \triangle t[-B_i^n + \alpha(3(r_i^n)^2 A_i^n + (s_i^n)^2 A_i^n + 2r_i^n s_i^n C_i^n)]$.
With the same process we discritize eqn(5.6) and we get

$$\begin{aligned}\frac{s^{n+1} - s^n}{\triangle t} + \alpha[3\frac{(s^2 r_x)^{n+1} + (s^2 s_x)^n}{2} + \frac{(r^2 s_x)^{n+1} + (r^2 s_x)^n}{2} \\ + 2\frac{(srr_x)^{n+1} + (srr_x)^n}{2}] + \frac{(s_{xxx})^{n+1} + (s_{xxx})^n}{2} = 0 \quad (5.10)\end{aligned}$$

Applying Rubin and Graves linearization technique[47] to linearize the non-linear terms, we get

$$\begin{aligned}2s^{n+1} + \triangle t[s_{3x}^{n+1} + 3\alpha((s^2)^n s_x^{n+1} + 2s^n s_x^n s_x^{n+1}) + \alpha((r^2)^n s_x^{n+1} + 2r^n s_x^{n+1} + 2r^n s_x^n r^{n+1}) \\ + 2\alpha(s^{n+1} r^n r_x^n + s^n r^{n+1} r_x^n + s^n r^n r_x^{n+1})] = 2s^n + \triangle t[-s_{3x}^n + 3\alpha(s^2)^n s_x^n \\ + \alpha(r^2)^n s_x^n + 2\alpha s^n r^n r_x^n] \quad (5.11)\end{aligned}$$

Substituting the terms A_i^n, B_i^n, C_i^n and D_i^n in eqn(5.11) we get

$$\begin{aligned}
&[2 + \triangle t(\tilde{w}_{ii}^{(3)} + \alpha(3(s_i^n)^2\tilde{w}_{ii}^{(1)} + 6s_i^n C_i^n + (r_i^n)^2\tilde{w}_{ii}^{(1)} \\
&\qquad + 2r_i^n A_i^n))]s_i^{n+1} + \sum_{j=0,j\neq i}^{N} \triangle t(\tilde{w}_{ij}^{(3)} + \alpha(3(s_i^n)^2\tilde{w}_{ij}^{(1)} \\
&\qquad + (r_i^n)^2\tilde{w}_{ij}^{(1)}))s_j + [2\alpha\triangle t(r_i^n C_i^n + s_i^n A_i^n + s_i^n r_i^n \\
&\qquad \tilde{w}_{ii}^{(1)})]r_i^{n+1} + [\sum_{j=0,j\neq i}^{N} (2\alpha\triangle t s_i^n r_i^n \tilde{w}_{ij}^{(1)})r_j^{n+1}] = g_i^n \qquad (5.12)
\end{aligned}$$

where $g_i = 2s_i^n + \triangle t[-D_i^n + \alpha(3(s_i^n)^2 C_i^n + (r_i^n)^2 C_i^n + 2s_i^n r_i^n A_i^n)]$

5.2 Implementation of DQM to KdV equation

Now, applying eqn(1.2) and boundary conditions to eqn(5.5) and eqn(5.6) we get

$$\frac{dr_i}{dt} = -\sum_{j=1}^{N-1} w_{ij}^{(3)} r_j - \alpha[2r_i s_i \sum_{j=1}^{N-1} w_{ij}^{(1)} s_j + (3r_i^2 + s_i^2)\sum_{j=1}^{N-1} w_{ij}^{(1)} r_j] \qquad (5.13)$$

$$\frac{ds_i}{dt} = -\sum_{j=1}^{N-1} w_{ij}^{(3)} s_j - \alpha[2r_i s_i \sum_{j=1}^{N-1} w_{ij}^{(1)} r_j + (3s_i^2 + r_i^2)\sum_{j=1}^{N-1} w_{ij}^{(1)} s_j] \qquad (5.14)$$

$i = 1, 2, \cdots, N-1$. The weighting coefficients $w_{ij}^{(1)}$ and $w_{ij}^{(3)}$ are determined eqn(3.28). To solve eqn(5.13) and eqn(5.14) we use RK4 method due to its advantages. For

CMKdVE, we consider three conserved quantities,

$$I_1 = \int_{-\infty}^{\infty} u dx$$
$$I_2 = \int_{-\infty}^{\infty} |u|^2 dx \simeq \sum_{j=1}^{N} h_j |u_j^n|^2$$
$$I_3 = \int_{-\infty}^{\infty} (\frac{\alpha}{2}|u|^4 - |u_x|^2) dx \simeq [\sum_{j=1}^{N} h_j \frac{\alpha}{2} |u_j^n|^4 - (u_x^n)_j^2]^{\frac{1}{2}}$$

where $h_j = x_j - x_{j-1}$ and u^n denotes the numerical solution at n^{th} time step, remain constant in time[42]. The accuracy of the method is measured by using the maximum error norm L_∞ presented as

$$L_\infty = ||U^{exact} - U^{num}||_\infty = max_j |U_j^{exact} - U_j^{num}|$$

5.3 Results and discussion

5.3.1 Single Soliton

The solution of CMKdV equation of the form[18],

$$U(x,t) = \sqrt{\frac{2c}{\alpha}} sech[\sqrt{c}(x - x_0 - ct)] exp(i\theta)$$

represents a single solitary wave of amplitude $\sqrt{2}c/\alpha$ moving to right with velocity c. We consider the parameters $\alpha = 2$, $\theta = 0$, $c = 1$, $x_0 = 0$ in the computational domain $[-20, 40]$ at $t = 0$ and obtained the initial condition. We applied MD-PDQ method to the equation and the maximum error norm is tabulated in Table 5.1. The error graph for single soliton is depicted in Figure5.1.

To show the efficiency of our present MD-PDQ method, we used QBDQM with the same parameters and the numerical results are tabulated in Table 5.2. Comparison of error with results of earlier numerical techniques is shown in Table 5.4.

5.3.2 Interaction of two Solitons

The interaction of two solitary waves with initial condition is give as[24],

$$U(x,0) = \sqrt{\frac{2c_1}{\alpha}} sech[\sqrt{c_2}(x - x_1)]exp(i\theta_1) + \sqrt{\frac{2c_2}{\alpha}} sech[\sqrt{c_2}(x - x_2)]exp(i\theta_2)$$

where $x_1 = 25$ and $x_2 = 50$ are the initial positions of the two solitary waves respectively in $[0, 100]$. We consider the parameters $\alpha = 2$, $c_1 = 2$, and $c_2 = 0.5$. We applied QBDQM methods to the given equation and the results thus obtained is tabulated in Table 5.3.

For complex modified KdV equation, the accuracy result of MD-PDQ is more because of the domain $[-20, 40]$ which is large. As the number of grid points increases it gave more better results than the previous methods as shown in Table 5.4.

Chapter 6

CONCLUSION

In this paper we implemented MD-PDQ method for the numerical solutions of KdV equation and complex modified KdV equation. For comparing we also use QBDQM and CDQM with same parameters. It can be observed from the results of the numerical experiments that MD-PDQ gave more accurate results in less time in comparing to other methods including QBDQM and CDQM. The performance and accuracy of the present method is shown by calculating and comparing L_∞ error norm with earlier works. In Table 4.4 and Table 5.4, we compared our present method with some earlier works, which shows that the present method produce more accurate numerical solution of the KdV equation and complex modified KdV equation respectively than previous methods. Three lowest invariants are calculated and are reported for both test problems. The obtained invariants are acceptable when compared with some earlier works. For high-dimensional non-linear KdV equations, the presented method can be applied. So, MD-PDQ is a reliable one for getting the numerical solutions of some physically important nonlinear problems.

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Table 4.1 : Error norms for Single Soliton for $\Delta t = 0.001$

Time	K=91 N=10, M=10 L_∞-error	K=136 N=10, M=15 L_∞-error
0	-	-
0.5	1.5276×10^{-6}	2.6254×10^{-7}
1.0	2.5323×10^{-6}	4.7215×10^{-7}
1.5	3.2232×10^{-6}	5.32541×10^{-7}
2.0	3.8236×10^{-6}	7.2425×10^{-7}

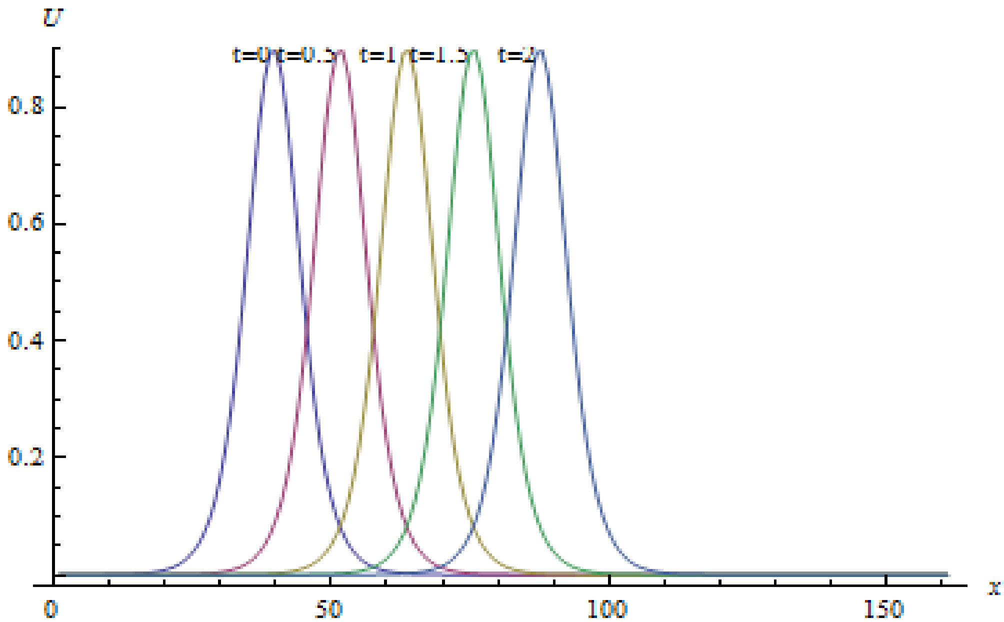

Figure 4.1 : Propagation of single soliton at t=0 to t=2.

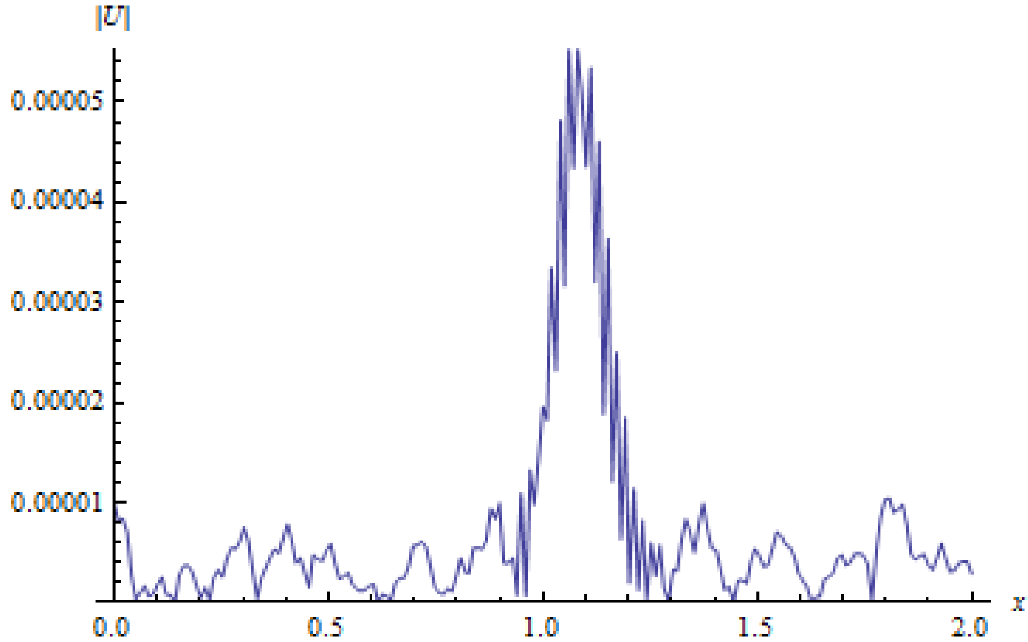

Figure 4.2 : Error norm at t = 2 for $\Delta t = 0.0001$ and step size, h = 0.01

Table 4.2 : Three invariants and error norm for single soliton for h= 0.01

Time	C_1	C_2	C_3	L_∞-error
0	0.144598	0.0867557	0.0468467	-
0.5	0.144599	0.0867557	0.0468482	9.02421×10^{-6}
1.0	0.144598	0.0867557	0.0468496	1.36957×10^{-5}
1.5	0.144597	0.0867558	0.0468546	3.02092×10^{-5}
2.0	0.144596	0.0867558	0.0468547	6.94995×10^{-5}

$\Delta t = 0.001$

Table 4.3 : Three invariants and error norm for a single soliton for h= 0.01

Time	C_1	C_2	C_3	$\boldsymbol{L_\infty}$-error
0	0.144598	0.0867575	0.0468456	0.000000
1.0	0.144598	0.0867579	0.0468485	0.060957
2.0	0.144598	0.086758	0.0468485	0.0552349

$\Delta t = 0.0001$

Table 4.4 : Comparison of error for Single soliton at t=1.0

Methods	h	Δt	L_∞-error
MD-PDQ(Present)	0.02	0.001	1.5273×10^{-6}
QBDQM(Present)	0.15	0.001	1.369×10^{-5}
CDQM(Present)	0.01	0.0001	6.095×10^{-5}
SCM*[34]	0.3	0.01	4.558×10^{-5}
MCB-DQM**[35]	0.01	0.001	106.9×10^{-5}
PDQ***[36]	0.02	0.001	274.5×10^{-5}

*sinc collocation Method

**Modified cubic B-spline Differential Quadrature Method

*** Polynomial based Differential Quadrature

Table 4.5 : Three invariants and error norm for interaction of two solitons for h= 0.15

Time	C_1	C_2	C_3	L_∞-error
-0.3	12	47.9774	211.048	-
-0.1	12	47.9821	211.145	0.00120921
0.1	11.9998	47.9822	211.145	0.00137692
0.3	11.9999	47.9768	211.041	0.00245652

$\Delta t = 0.001$

Table 4.6 : Three invariants and error norm for interaction of two solitons for h=0.15

Time	C_1	C_2	C_3	L_∞-error
-0.3	12	47.9953	211.173	-
-0.1	12	47.9965	211.131	0.00276263
0.1	12.0001	47.9965	211.153	0.00256739
0.3	12	47.9953	211.203	0.000318579

Δt =0.00001

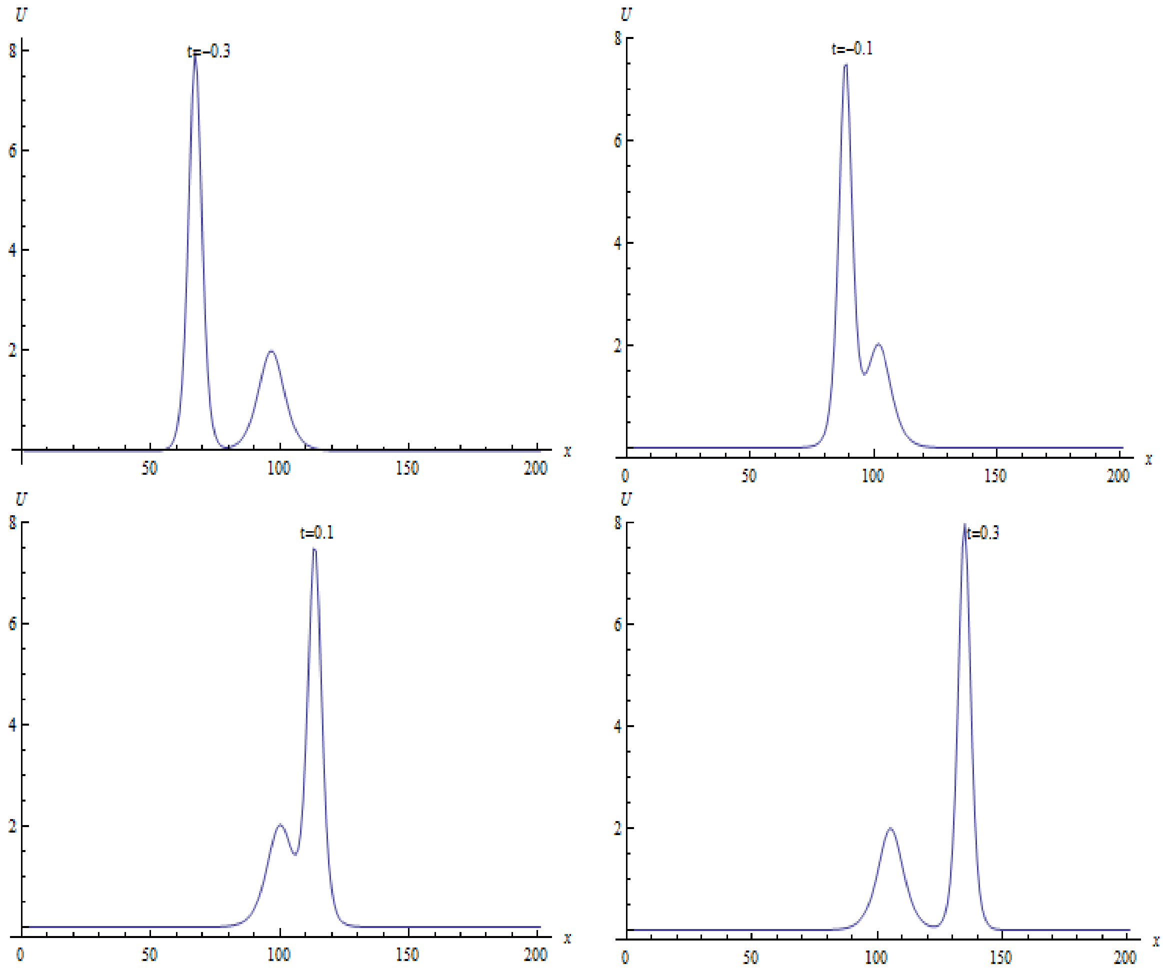

Figure 4.3 : Interaction of two solitons at t=-0.3, t=-0.1, t=0.1, t=0.3.

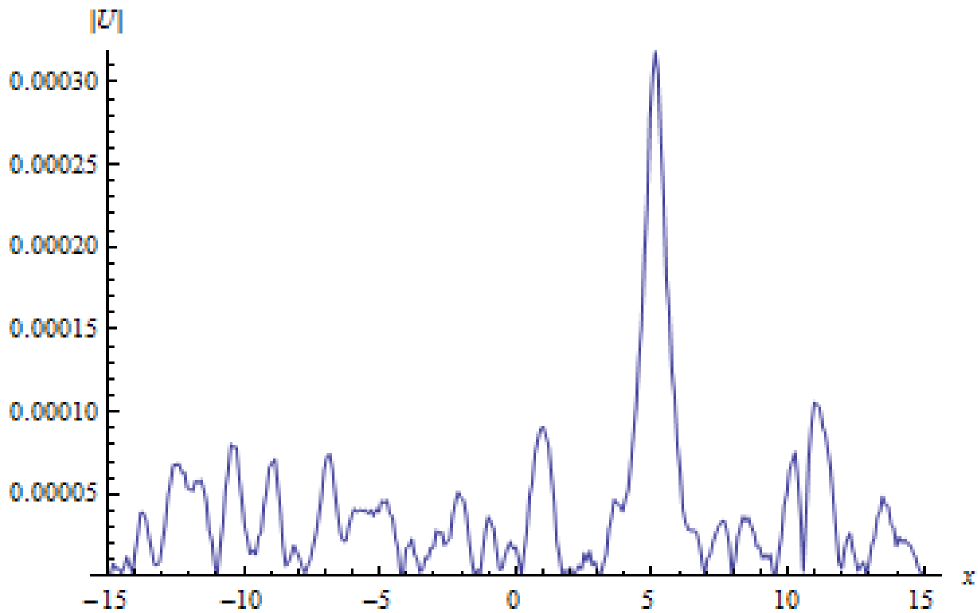

Figure 4.4 : Error norm at t = 0.3 for $\Delta t = 0.00001$ and step size, h = 0.1

Table 4.7 : Comparison of error for Interaction of two solitons at t=0.1

Methods	h	Δt	L_∞-error
QBDQM(Present)	0.15	0.001	1.38×10^{-3}
CDQM(Present)	0.1	0.00001	2.57×10^{-3}
MBQI*[37]	0.1	0.00001	3.84×10^{-3}
MQQI**[38]	0.1	0.00001	7.74×10^{-3}
RBF(MQ)***[39]	0.1	0.00001	9.21×10^{-4}
RBF(IMQ)****[39]	0.1	0.00001	2.21×10^{-2}

*Multilevel B-spline quasi-interpolation

**Multiquadric quasi-interpolation ***Radial basis function (Multiquadrics)

****Radial basis function(Inverse multiquadrics)

Table 5.1 : Three invariants and error norm for Single Soliton for $\Delta t = 0.001$

Time	K=326 N=10, M=40 L_∞-error	I_1	I_2	I_3
0	-	3.14159	2.00000	0.66667
4	2.7562×10^{-7}	3.14158	1.99999	0.66667
6	3.6234×10^{-7}	3.14158	1.99999	0.66667
8	4.7232×10^{-7}	3.14157	1.99999	0.66667
10	6.1532×10^{-7}	3.14157	1.99999	0.66667

Table 5.2 : Three invariants and error norm for a single soliton for h=0.2

Time	I_1	I_2	I_3	L_∞-error
0	3.1416	1.99996	0.6666	-
1	3.14157	1.99995	0.66677	2.91873×10^{-6}
2	3.1416	1.99995	0.66666	3.13024×10^{-6}
3	3.1416	1.99995	0.66666	4.34549×10^{-6}
4	3.1416	1.99995	0.66677	5.37757×10^{-6}
5	3.1416	1.99996	0.66669	6.09012×10^{-6}

$\Delta t = 0.001$

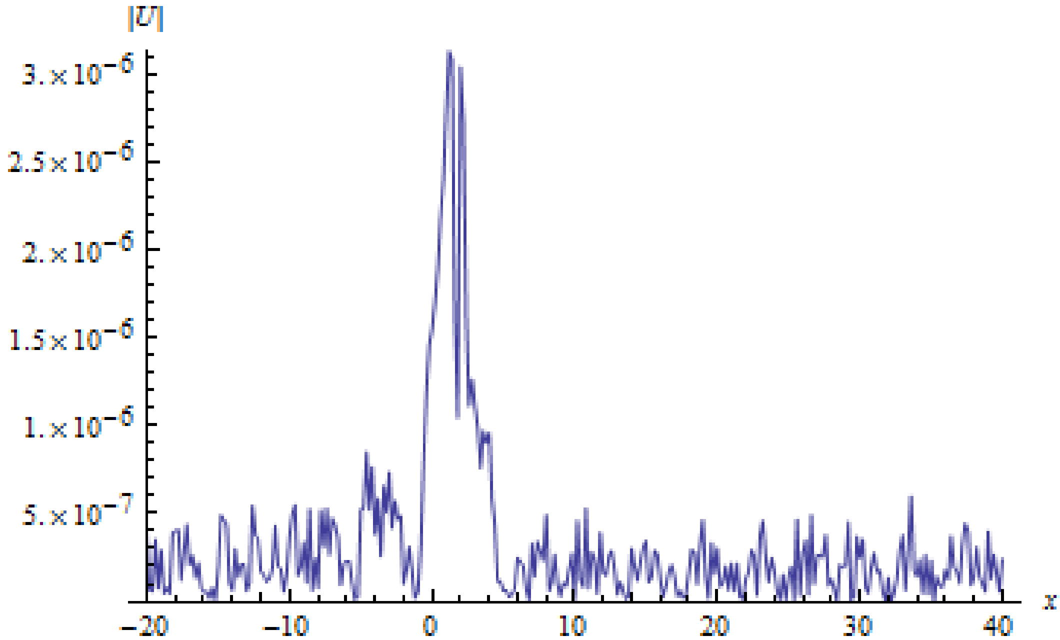

Figure 5.1 : Error norm at t = 2 for $\Delta t = 0.001$ and step size h=0.2

Table 5.3 : Invariants of interaction of two solitons at $\Delta t = 0.01$

Time	I_1(Re)	I_1(Im)	I_2	I_3
0	3.14158	3.14150	4.24079	2.11544
1	3.15585	3.15549	4.24096	2.11528
2	3.16348	3.16349	4.24097	2.1147
3	3.16876	3.16877	4.24118	2.11369
4	3.17403	3.17391	4.24149	2.11315
5	3.19474	3.19475	4.24182	2.10902

Table 5.4 : Comparison of error for single soliton at t = 5

Methods	h	Δt	L_∞-error
MD-PDQ(Present)	0.15	0.001	3.1898×10^{-7}
QBDQM(Present)	0.2	0.001	6.09012×10^{-6}
QBCM*[41]	0.05	0.005	0.03056×10^{-3}
PGM**[47]	0.05	0.001	5.7×10^{-5}
FD***[48]	0.5	0.0125	0.4×10^{-5}

*Quintic B-spline Collocation Method

**Petrov- Galerkin Method

***Finite Difference scheme